# Error calculus and regularity of Poisson functionals: the lent particle method.

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**Abstract**

We propose a new method to apply the Lipschitz functional calculus of local Dirichlet forms to Poisson random measures.

**Résumé**

**Calcul d'erreur et régularité des fonctionnelles de Poisson : la méthode de la particule prêtée.** Nous proposons une nouvelle méthode pour appliquer le calcul fonctionnel lipschitzien des formes de Dirichlet locales aux mesures aléatoires de Poisson.

## 1 Notation and basic formulae.

Let us consider a local Dirichlet structure with carré du champ $(X, \mathcal{X}, \nu, \mathbf{d}, \gamma)$ where $(X, \mathcal{X}, \nu)$ is a $\sigma$-finite measured space called *bottom-space*. Singletons are in $\mathcal{X}$ and $\nu$ is diffuse, $\mathbf{d}$ is the domain of the Dirichlet form $\epsilon[u] = 1/2 \int \gamma[u] d\nu$. We denote $(a, \mathcal{D}(a))$ the generator in $L^2(\nu)$ (cf. [3]).

A random Poisson measure associated to $(X, \mathcal{X}, \nu)$ is denoted $N$. $\Omega$ is the configuration space of countable sums of Dirac masses on $X$ and $\mathcal{A}$ is the $\sigma$-field generated by $N$, of law $\mathbb{P}$ on $\Omega$. The space $(\Omega, \mathcal{A}, \mathbb{P})$ is called *the up-space*. We write $N(f)$ for $\int f dN$. If $p \in [1, \infty[$ the set $\{e^{iN(f)} : f \text{ real}, f \in L^1 \cap L^2(\nu)\}$ is total in $L^p_{\mathbb{C}}(\Omega, \mathcal{A}, \mathbb{P})$. We put $\tilde{N} = N - \nu$. The relation $\mathbb{E}(\tilde{N} f)^2 = \int f^2 d\nu$ extends and gives sense to $\tilde{N}(f)$, $f \in L^2(\nu)$. The Laplace functional and the differential calculus with $\gamma$ yield

$$(1) \qquad \forall f \in \mathbf{d}, \forall h \in \mathcal{D}(a) \qquad \mathbb{E}[e^{i\tilde{N}(f)}(\tilde{N}(a[h]) + \frac{i}{2} N(\gamma[f, h])] = 0.$$

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## 2 Product, particle by particle, of a Poisson random measure by a probability measure.

Given a probability space $(R,\mathcal{R},\rho)$, let us consider a Poisson random measure $N\odot\rho$ on $(X\times R,\mathcal{X}\times\mathcal{R})$ with intensity $\nu\times\rho$ such that for $f\in L^1(\nu)$ and $g\in L^1(\rho)$ if $N(f)=\sum f(x_n)$ then $(N\odot\rho)(fg)=\sum f(x_n)g(r_n)$ where the $r_n$'s are i.i.d. independent of $N$ with law $\rho$. Calling $(\hat{\Omega},\hat{\mathcal{A}},\hat{\mathbb{P}})$ the product of all the factors $(R,\mathcal{R},\rho)$ involved in the construction of $N\odot\rho$, we obtain the following properties : For an $\mathcal{A}\times\mathcal{X}\times\mathcal{R}$-measurable and positive function $F$, $\hat{\mathbb{E}}\int F(\omega,x,r)N\odot\rho(dxdr)=\int F\,d\rho\,dN\quad\mathbb{P}$-a.s.

Let us denote by $\mathbb{P}_N$ the measure $\mathbb{P}(d\omega)N_\omega(dx)$ on $(\Omega\times X,\mathcal{A}\times\mathcal{X})$. We have

**Lemma 2.1** *Let $F$ be $\mathcal{A}\times\mathcal{X}\times\mathcal{R}$-measurable, $F\in\mathcal{L}^2(\mathbb{P}_N\times\rho)$ and such that*
$\int F(\omega,x,r)\,\rho(dr)=0\quad\mathbb{P}_N$*-a.s., then $\int F\,d(N\odot\rho)$ is well defined, belongs to $L^2(\mathbb{P}\times\hat{\mathbb{P}})$ and*

$$(2)\qquad \hat{\mathbb{E}}\Big(\int F\,d(N\odot\rho)\Big)^2=\int F^2\,dN\,d\rho\qquad\mathbb{P}\text{-}a.s.$$

The argument consists in considering $F_n$ satisfying
$\mathbb{E}\int F_n^2\,d\nu d\rho<+\infty$ and $\mathbb{E}\int(\int|F_n|\,d\nu)^2d\rho<+\infty$ and then using the relation
$\hat{\mathbb{E}}(\int F_n\,d(N\odot\rho))^2=(\int F_n d\rho dN)^2-\int(\int F_n d\rho)^2dN+\int F_n^2 d\rho dN\quad\mathbb{P}$-a.s.

## 3 Construction by Friedrichs' method and expression of the gradient.

a) We suppose the space by $\mathbf{d}$ of the bottom structure is separable, then a gradient exists (cf. [3] Chap. V, p.225 *et seq.*). We denote it $\flat$ and choose it with values in the space $L^2(R,\mathcal{R},\rho)$. Thus, for $u\in\mathbf{d}$ we have $u^\flat\in L^2(\nu\times\rho)$, $\gamma[u]=\int(u^\flat)^2d\rho$ and $\flat$ satisfies the chain rule. We suppose in addition, what is always possible, that $\flat$ takes its values in the subspace orthogonal to the constant 1, i.e.

$$(3)\qquad \forall u\in\mathbf{d}\qquad \int u^\flat\,d\rho=0\quad\nu\text{-a.s.}$$

This hypothesis is important here as in many applications (cf. [2] Chap V §4.6). We suppose also, but this is not essential (cf. [3] p44) $1\in\mathbf{d}_{loc}\quad\gamma[1]=0$ so that $1^\flat=0$.

b) We define a pre-domain $\mathcal{D}_0$ dense in $L^2_{\mathbb{C}}(\mathbb{P})$ by

$$\mathcal{D}_0=\{\sum_{p=1}^m\lambda_p e^{i\tilde{N}(f_p)};m\in\mathbb{N}^*,\lambda_p\in\mathbb{C},f_p\in\mathcal{D}(a)\cap L^1(\nu)\}.$$

c) We introduce the creation operator inspired from quantum mechanics (see [7], [8], [9], [1], [5],[6] and [10] among others) defined as follows

$$(4)\qquad \varepsilon_x^+(\omega)\text{ equals }\omega\text{ if }x\in\operatorname{supp}(\omega),\quad\text{and equals }\omega+\varepsilon_x\text{ if }x\notin\operatorname{supp}(\omega)$$

so that

$$(5)\qquad \varepsilon_x^+(\omega)=\omega\quad N_\omega\text{-a.e. }x\qquad\text{and}\qquad\varepsilon_x^+(\omega)=\omega+\varepsilon_x\quad\nu\text{-a.e. }x$$

This map is measurable and the Laplace functional shows that for an $\mathcal{A}\times\mathcal{X}$-measurable $H \geq 0$,

$$\mathbb{E}\int \varepsilon^+ H\, d\nu = \mathbb{E}\int H\, dN. \tag{6}$$

Let us remark also that by (5), for $F \in \mathcal{L}^2(\mathbb{P}_N \times \rho)$

$$\int \varepsilon^+ F\, d(N \odot \rho) = \int F d(N \odot \rho) \qquad \mathbb{P}\times\hat{\mathbb{P}}\text{-a.s.} \tag{7}$$

d) We defined a gradient $\sharp$ for the up-structure on $\mathcal{D}_0$ by putting for $F \in \mathcal{D}_0$

$$F^\sharp = \int (\varepsilon^+ F)^\flat\, d(N \odot \rho) \tag{8}$$

this definition being justified by the fact that for $\mathbb{P}$-a.e. $\omega$ the map $y \mapsto F(\varepsilon_y^+(\omega)) - F(\omega)$ is in $\mathbf{d}$, $\varepsilon^+ F$ belongs to $L^\infty(\mathbb{P}) \otimes \mathbf{d}$ algebraic tensor product, and $(\varepsilon^+ F - F)^\flat = (\varepsilon^+ F)^\flat \in L^2(\mathbb{P}_N \times \rho)$.

For $F, G \in \mathcal{D}_0$ of the form

$$F = \sum_p \lambda_p e^{i\tilde{N}(f_p)} = \Phi(\tilde{N}(f_1), \ldots, \tilde{N}(f_m)) \qquad G = \sum_q \mu_q e^{i\tilde{N}(g_q)} = \Psi(\tilde{N}(g_1), \ldots, \tilde{N}(g_n))$$

we compute using (2), (3) and (7) (in the spirit of prop. 1 of [9] or lemma 1.2 of [6])

$$\hat{\mathbb{E}}[F^\sharp \overline{G^\sharp}] = \sum_{p,q} \lambda_p \overline{\mu_q} e^{i\tilde{N}(f_p) - i\tilde{N}(g_q)} \gamma[f_p, g_q] \tag{9}$$

and we have

**Proposition 3.1** *If we put* $A_0[F] = \sum_p \lambda_p e^{i\tilde{N}(f_p)}(i\tilde{N}(a[f_p]) - \frac{1}{2}N(\gamma[f_p]))$ *it comes*

$$\mathbb{E}[A_0[F]\overline{G}] = -\frac{1}{2}\mathbb{E}\sum_{p,q} \Phi'_p \overline{\Psi'_q} N(\gamma[f_p, g_q]). \tag{10}$$

In order to show that $A_0[F]$ does not depend on the form of $F$, by (10) it is enough to show that the expression $\sum_{p,q} \Phi'_p \overline{\Psi'_q} N(\gamma[f_p, g_q])$ depends only on $F$ and $G$. But this comes from (9) since $F^\sharp$ and $G^\sharp$ depend only on $F$ and $G$.

By this proposition, $A_0$ is symmetric on $\mathcal{D}_0$, negative, and the argument of Friedrichs applies (cf [3] p4), $A_0$ extends uniquely to a selfadjoint operator $(A, \mathcal{D}(A))$ which defines a closed positive (hermitian) quadratic form $\mathcal{E}[F] = -\mathbb{E}[A[F]\overline{F}]$. By (10) contractions operate and (cf. [3]) $\mathcal{E}$ is a Dirichlet form which is local with carré du champ denoted $\Gamma$ and the up-structure obtained $(\Omega, \mathcal{A}, \mathbb{P}, \mathbb{D}, \Gamma)$ satisfies

$$\forall f \in \mathbf{d}, \quad \tilde{N}(f) \in \mathbb{D} \text{ and } \Gamma[\tilde{N}(f)] = N(\gamma[f]) \tag{11}$$

The operator $\sharp$ extends to a gradient for $\Gamma$ as a closed operator from $L^2(\mathbb{P})$ into $L^2(\mathbb{P} \times \hat{\mathbb{P}})$ with domain $\mathbb{D}$ which satisfies the chain rule and may be computed on functionals $\Phi(\tilde{N}(f_1), \ldots, \tilde{N}(f_m))$, $\Phi$ Lipschitz and $\mathcal{C}^1$ and their limits in $\mathbb{D}$ (as done in [4]).

Formula (8) for $\sharp$ can be extended from $\mathcal{D}_0$ to $\mathbb{D}$. Let us introduce the space $\underline{\mathbb{D}}$ closure of $\mathcal{D}_0 \otimes \mathbf{d}$ for the norm

$$\|H\|_{\underline{\mathbb{D}}} = (\mathbb{E}\int \gamma[H(\omega,.)](x)\ N(dx))^{1/2} + \mathbb{E}\int |H(\omega,x)|\xi(x)\ N(dx)$$

where $\xi > 0$ is a fixed function such that $N(\xi) \in L^2(\mathbb{P})$.

**Theorem 3.1** *The formula $F^\sharp = \int (\varepsilon^+ F)^\flat\ d(N \odot \rho)$ decomposes as follows*

$$F \in \mathbb{D} \quad \xrightarrow{\varepsilon^+} \quad \varepsilon^+ F \in \underline{\mathbb{D}} \quad \xrightarrow{\flat} \quad (\varepsilon^+ F)^\flat \in L^2_0(\mathbb{P}_N \times \rho) \quad \xrightarrow{d(N\odot\rho)} \quad F^\sharp \in L^2(\mathbb{P} \times \hat{\mathbb{P}})$$

*where each operator is continuous on the range of the preceding one, $L^2_0(\mathbb{P}_N \times \rho)$ denoting the closed subspace of $L^2(\mathbb{P}_N \times \rho)$ of $\rho$-centered elements, and we have*

$$\Gamma[F] = \hat{\mathbb{E}}|F^\sharp|^2 = \int \gamma[\varepsilon^+ F]\ dN. \tag{12}$$

# 4 The lent particle method.

Let us consider, for instance, a real process $Y_t$ with independent increments and Lévy measure $\sigma$ integrating $x^2$, $Y_t$ being supposed centered without Gaussian part. We assume that $\sigma$ has an l.s.c. density so that a local Dirichlet structure may be constructed on $\mathbb{R}\backslash\{0\}$ with carré du champ $\gamma[f] = x^2 f'^2(x)$. If $N$ is the random Poisson measure with intensity $dt \times \sigma$ we have $\int_0^t h(s)\ dY_s = \int 1_{[0,t]}(s)h(s)x\tilde{N}(dsdx)$ and the choice done for $\gamma$ gives $\Gamma[\int_0^t h(s)dY_s] = \int_0^t h^2(s)d[Y,Y]_s$ for $h \in L^2_{loc}(dt)$. In order to study the regularity of the random variable $V = \int_0^t \varphi(Y_{s-})dY_s$ where $\varphi$ is Lipschitz and $\mathcal{C}^1$, we have two ways:

a) We may represent the gradient $\sharp$ as $Y_t^\sharp = B_{[Y,Y]_t}$ where $B$ is a standard auxiliary independent Brownian motion. Then by the chain rule $V^\sharp = \int_0^t \varphi'(Y_{s-})(Y_{s-})^\sharp dY_s + \int_0^t \varphi(Y_{s-})dB_{[Y]_s}$ now, using $(Y_{s-})^\sharp = (Y_s^\sharp)_-$, a classical but rather tedious stochastic computation yields

$$\Gamma[V] = \hat{\mathbb{E}}[V^{\sharp 2}] = \textstyle\sum_{\alpha \le t} \Delta Y_\alpha^2 (\int_{]\alpha}^t \varphi'(Y_{s-})dY_s + \varphi(Y_{\alpha-}))^2. \tag{13}$$

Since $V$ has real values the *energy image density property* holds, and $V$ has a density as soon as $\Gamma[V]$ is strictly positive a.s. what may be discussed using the relation (13).

b) Another more direct way consists in applying the theorem. For this we define $\flat$ by choosing $\eta$ such that $\int_0^1 \eta(r)dr = 0$ and $\int_0^1 \eta^2(r)dr = 1$ and putting $f^\flat = xf'(x)\eta(r)$.

$1^o$. First step. We add a particle $(\alpha, x)$ i.e. a jump to $Y$ at time $\alpha$ with size $x$ what gives
$\varepsilon^+ V - V = \varphi(Y_{\alpha-})x + \int_{]\alpha}^t (\varphi(Y_{s-} + x) - \varphi(Y_{s-}))dY_s$

$2^o$. $V^\flat = 0$ since $V$ does not depend on $x$, and
$(\varepsilon^+ V)^\flat = (\varphi(Y_{\alpha-})x + \int_{]\alpha}^t \varphi'(Y_{s-} + x)xdY_s)\eta(r)$ because $x^\flat = x\eta(r)$.

$3^o$. We compute $\gamma[\varepsilon^+ V] = \int (\varepsilon^+ V)^{\flat 2} dr = (\varphi(Y_{\alpha-})x + \int_{]\alpha}^t \varphi'(Y_{s-} + x)xdY_s)^2$

$4^o$. We take back the particle we gave, because in order to compute $\int \gamma[\varepsilon^+ V]dN$ the integral in $N$ confuses $\varepsilon^+\omega$ and $\omega$.

That gives $\int \gamma[\varepsilon^+ V]dN = \int (\varphi(Y_{\alpha-}) + \int_{]\alpha}^t \varphi'(Y_{s-})dY_s)^2 x^2 \; N(d\alpha dx)$ and (13).

We remark that both operators $F \mapsto \varepsilon^+ F$, $F \mapsto (\varepsilon^+ F)^\flat$ are non-local, but instead $F \mapsto \int (\varepsilon^+ F)^\flat \, d(N \odot \rho)$ and $F \mapsto \int \gamma[\varepsilon^+ F] \, dN$ are local : taking back the lent particle gives the locality.

# References


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